Combinatorial contemplations

Sergey Kirgizov

Habilitation à Diriger des Recherches
LIB, Université Bourgogne Europe

Composition du Jury :

Antoine Genitrini, Professeur à Sorbonne Université	Rapporteur
Renzo Pinzani, Professeur émérite à l'Université de Florence	Rapporteur
Eric Rivals, Directeur de Recherches CNRS, Université de Montpellier	Rapporteur
Olivier Bodini, Professeur à l'Université Sorbonne Paris Nord	Examinateur
Enrica Duchi, Professeure à l'Université Paris-Cité	Examinatrice
Alain Giorgetti, Professeur à l'Université Marie et Louis Pasteur	Examinateur
Elena Barcucci, Ancienne Professeure à l'Université de Florence	Invitée

16 March 2026

My path to combinatorics

1988 - Born in Tselinograd, Kazakhstan, USSR
1992 - Tselinograd is renamed to Akmola
                   (renamed to Astana in 1998)
1993 - Our family moved from Akmola to Barnaul
2009 - B.S. in Mathematics, Altay State University,
           Barnaul, Russia
2011 - Master in Applied Mathematics and Informatics,
           Altay State University
2014 - PhD in Université Pierre et Marie Curie
2014-2019 Postdoc & industry positions
from 2019 - Maître de conférences,
                   Université Bourgogne Europe

My recent work in Dijon
is mostly about the structure and enumeration of discrete
objects: words, paths, patterns, permutations, …

In 2023, we organized Permutation Patterns conference.

PhD Students

- Abdelhamid Garah (co-supervised with Nader Mbarek), 2021-2024
- Rémi Maréchal (with Jean-Luc Baril and Vincent Vajnovszki), 2021-2024
- Nathanaël Hassler (with JLB and VV), 2023-ongoing

Postdocs

- Célia Biane, 2023-2024
- Khaydar Nurligareev, 2023-2024

Master intern

- Daniel Pinson, 2024

Project leader

- ANR "PICS — Patterns in combinatorics", 2022-ongoing
- "Artico — L'art de la combinatoire et ses interactions", Bourgogne-Franche-Comté region, 2021-2023

Participant in other projects: ANR COREGRAPHIE, ANR COMETA-GAE, …

Enumeration via generating functions

Generating functions

Binary alphabet: α and β.
How many words of size n do we have?

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$\boxed{\varepsilon}$

where ε denotes the unique empty word.

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| ε | α β | $\alpha\alpha$ $\alpha\beta$ $\beta\alpha$ $\beta\beta$ |

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Cardinality sequence: 1, 2, 4, 8, 16,...

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Cardinality sequence: 1, 2, 4, 8, 16,...
How can this infinite sequence be written in a finite way?

It can be written as

$$\frac{1}{1 - 2x}$$

But why?

Generating functions

Denote the union by $+$. The set $\mathcal{W}$ of binary words is

$$\mathcal{W} = \varepsilon + \alpha + \beta + \alpha\alpha + \alpha\beta + \beta\alpha + \beta\beta + \alpha\alpha\alpha + \alpha\alpha\beta \ldots$$

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$\mathcal{W}\alpha$ — append α to all words from $\mathcal{W}$.
(Same for $\mathcal{W}\beta$ — append β).

Observe that

$$\mathcal{W} = \varepsilon + \mathcal{W}\alpha + \mathcal{W}\beta$$

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Observe that

$$\mathcal{W} = \varepsilon + \mathcal{W}\alpha + \mathcal{W}\beta$$

Treat concatenation as multiplication. Replace ε by 1.
Denote the result by W.

$$W = 1 + \alpha + \beta + \alpha^2 + 2\alpha\beta + \beta^2 + \alpha^3 + 3\alpha^2\beta + \ldots$$

$$W = 1 + W\alpha + W\beta$$

Generating functions

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Treat W as a function of α and β.

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Taylor expansion of $W(\alpha, \beta)$ around $(0, 0)$ gives

$$\sum_{m=0, k=0}^{\infty} b_{m,k} \alpha^m \beta^k,$$

where $b_{m,k}$ is the number of binary words of size $m + k$, having m αs and k βs.

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where $b_{m,k}$ is the number of binary words of size $m + k$, having m αs and k βs.

Calculate $W(x, x) = \dfrac{1}{1 - 2x}$. Its Taylor expansion, $\displaystyle\sum_{n=0}^{\infty} b_n x^n$,

gives the sequence b_n, the number of binary words of size n (of course $b_n = 2^n$).

Classical Goulden-Jackson approach

Consecutive patterns in binary words

How many binary words of size *n* have exactly *k* occurrences of a consecutive pattern 01010 ?

Consecutive patterns in binary words

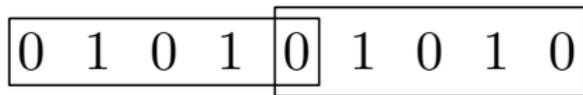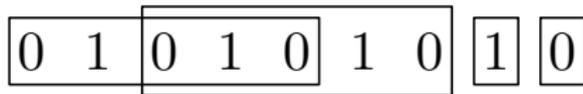How many binary words of size n have exactly k occurrences of a consecutive pattern 01010 ?

Idea: construct words using 0, 1 and 01010.

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Problems: duplicates and overlaps.
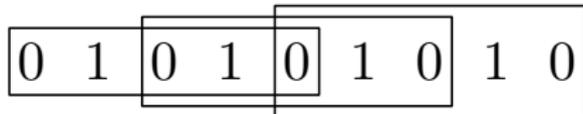
Same words with different decompositions.
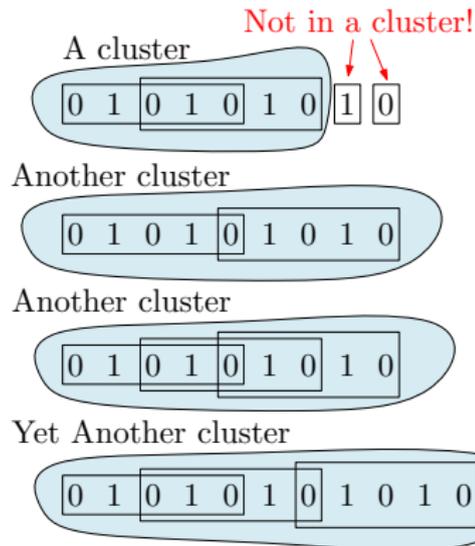
Autocorrelations and clusters of patterns

01010 can overlap with itself in several ways:

```
01010          01010          01010
01010              01010              01010
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Its autocorrelation polynomial is $A_{01010} = 1 + x^2 + x^4$.

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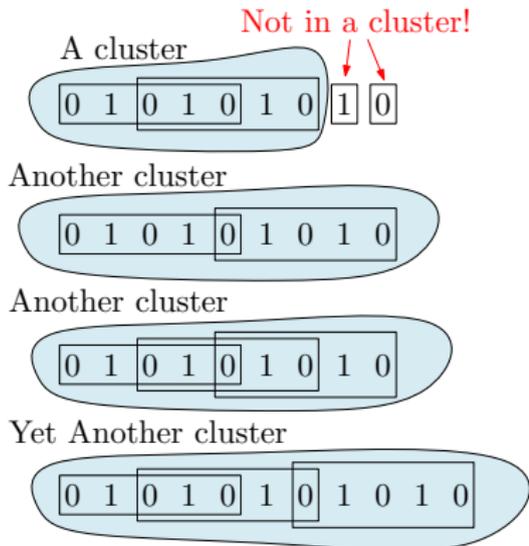Overlapping occurrences of a pattern form a *cluster*.

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Generating function for clusters of labelled overlapping patterns

$$C_{01010}(x, u) = \frac{ux^5}{1 - u(x^2 + x^4)},$$

where x is for size and u is for labelled patterns.

Symbolic inclusion-exclusion principle

Imagine we have a g.f. for a distribution of a given pattern μ:

$$B_\mu(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} d_{n,k} x^n y^k.$$

There are $d_{n,k}$ words of size n with k occurrences of μ.

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We label *some* occurrences of μ by the variable u,
i.e. y is replaced either by 1 (no label) or by u (label).

$$L_\mu(x, u) = B_\mu(x, 1 + u).$$

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It is simpler to construct $L_\mu(x, u)$ than $B_\mu(x, y)$!
Once we have $L_\mu(x, u)$, we recover $B_\mu(x, y)$ by *the symbolic inclusion-exclusion*: $B_\mu(x, y) = L_\mu(x, y - 1)$

For binary words and a pattern μ we get:

$$B_\mu(x, y) = L_\mu(x, y - 1),$$

$$L_\mu(x, u) = \frac{1}{1 - 2x - C_\mu(x, u)},$$

where $C_\mu(x)$ is the g.f. for clusters of overlapping labelled occurrences of μ calculated using its autocorrelation polynomial $A_\mu(x)$

$$C_\mu(x, u) = \frac{ux^{|\mu|}}{1 - u(A_\mu(x) - 1)}.$$

For a pattern μ, A_μ contains x^j iff two occurrences can overlap at $|\mu| - j$ letters.

Also possible for joint distribution of several patterns.

x for size, u for labelled occurrences of 01010.

$$\underline{01010}1010 \mapsto x^9 u$$
$$+$$
$$01\underline{01010}10 \mapsto x^9 u$$
$$+$$
$$0101\underline{01010} \mapsto x^9 u$$
$$+$$
$$\underline{010\underline{01010}} \mapsto x^9 u^2$$
$$+$$
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$$01\underline{\underline{01010}} \mapsto x^9 u^2$$
$$+$$
$$\underline{\underline{\underline{01010}}} \mapsto x^9 u^3$$
$$+$$
$$010101010 \mapsto x^9$$

13

x for size, u for labelled occurrences of 01010.

$$
\left.
\begin{array}{l}
\underline{010\underline{10}}1010 \mapsto x^9 u \\
\quad + \\
010\underline{101}010 \mapsto x^9 u \\
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\quad + \\
010101010 \mapsto x^9
\end{array}
\right\}
\implies
\left.
\begin{array}{l}
\underline{010\underline{10}}1010 \mapsto x^9(y-1) \\
\quad + \\
010\underline{101}010 \mapsto x^9(y-1) \\
\quad + \\
0101\underline{01010} \mapsto x^9(y-1) \\
\quad + \\
\underline{010\underline{10}1010} \mapsto x^9(y-1)^2 \\
\quad + \\
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\quad + \\
010101010 \mapsto x^9
\end{array}
\right\}
$$

13

x for size, u for labelled occurrences of 01010.

$$
\left.
\begin{aligned}
\underline{010101010} &\mapsto x^9 u \\
&+ \\
010\underline{10101}0 &\mapsto x^9 u \\
&+ \\
01010\underline{10101}0 \ \text{-like} &\mapsto x^9 u \\
&+ \\
\underline{010\underline{10101}0} &\mapsto x^9 u^2 \\
&+ \\
\underline{0101\underline{0101}0} &\mapsto x^9 u^2 \\
&+ \\
010\underline{10101}0 &\mapsto x^9 u^2 \\
&+ \\
\underline{010\underline{10101}0} &\mapsto x^9 u^3 \\
&+ \\
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&+ \\
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\end{aligned}
\right\}
\implies \underline{\underline{010101010}} \mapsto x^9 y^3 .
$$

Endhered patterns in RNA

Inspiration: absent patterns in ADN

No occurrences of these patterns in the human genome	CGCTCGACGTA, GTCCGAGCGTA, CGACGAACGGT, CCGATACGTCG

✎ *Absent sequences: nullomers and primes, 2007*
by Greg Hampikian and Tim Andersen

RNA secondary structures and matchings
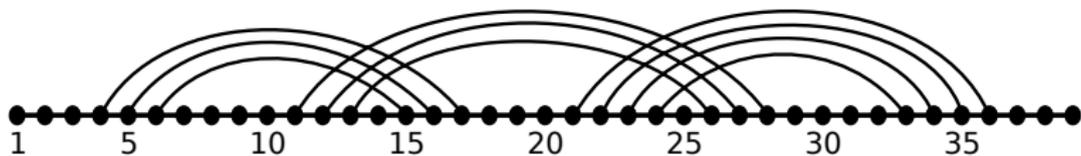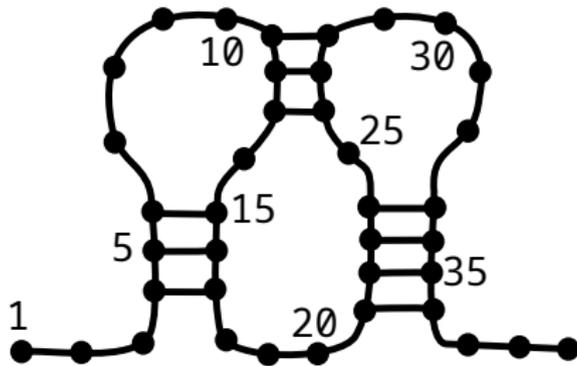
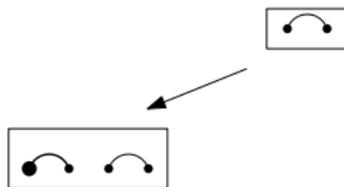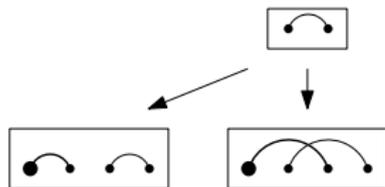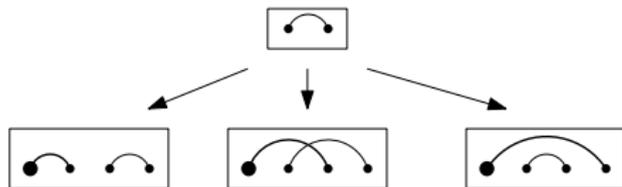$$...(((\ldots[[[.)))\ldots((((.]]]....))))...$$

Perfect matchings

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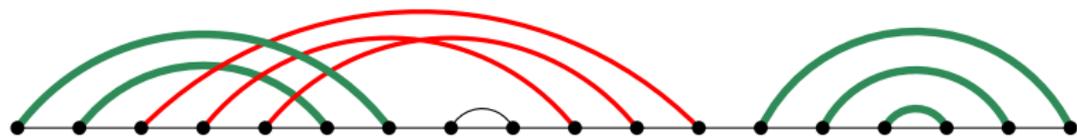

$2(n-1)$ old points
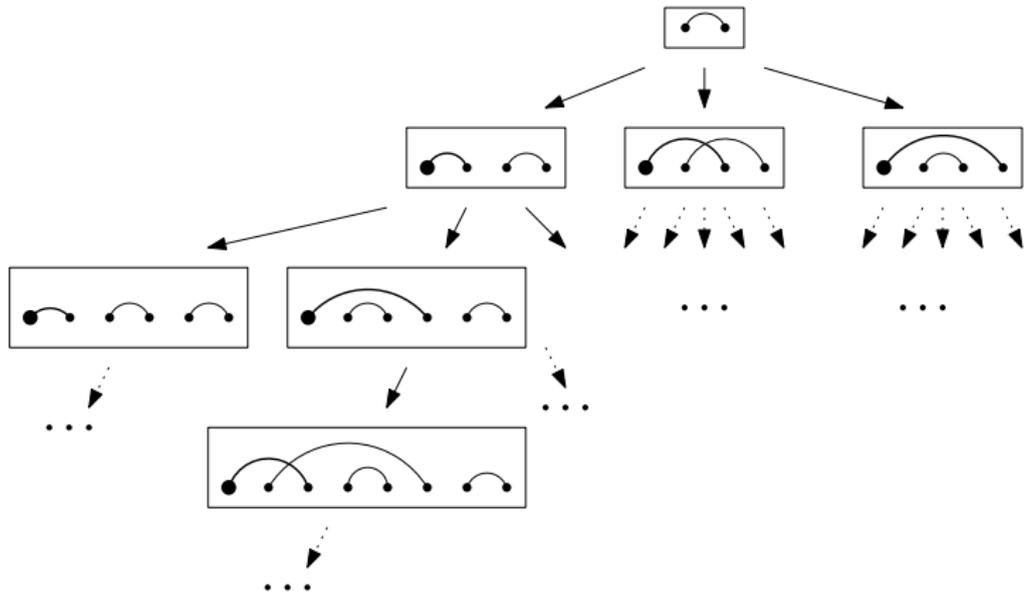
$2(n-1)+1$ possible new arcs endings
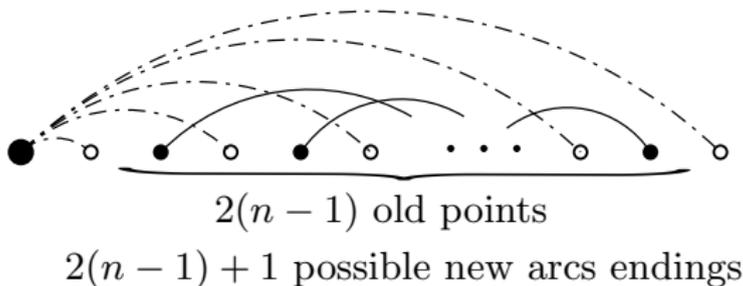
$a_n = (2n-1) \cdot (2n-3) \cdots 5 \cdot 3 \cdot 1 = (2n-1)!!$

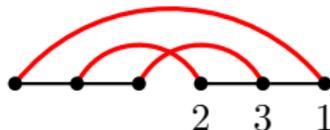A1147 in Sloane's Encyclopedia : 1, 3, 15, 105, 945, 10395,…
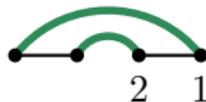
Endhered patterns in perfect matchings

(endhered = **end**-ad**hered**)

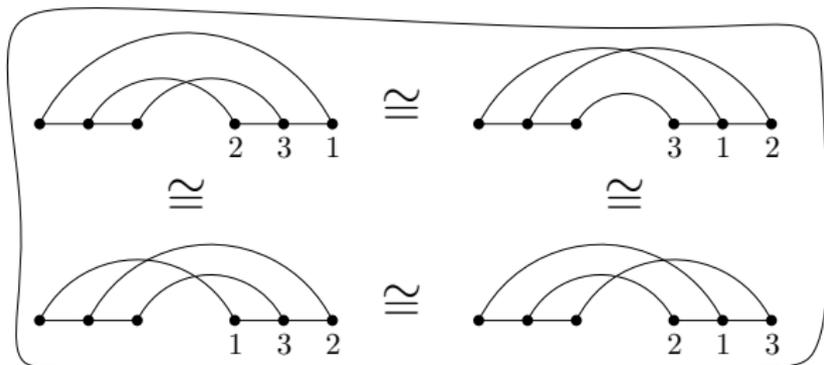This matching contains
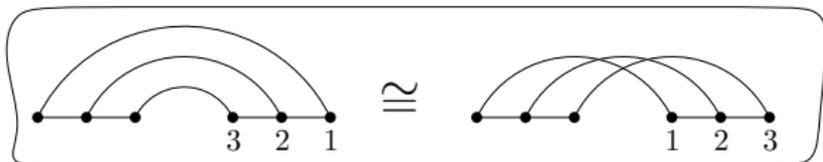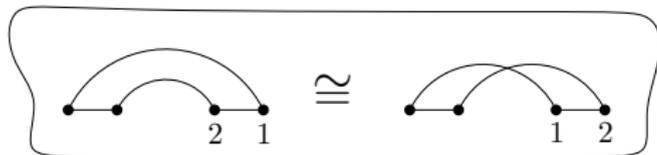
1 occurrence of pattern

3 occurrences of pattern

We write patterns in condensed form, indicating sequentially
the order of starting points corresponding to arc ends.

Endhered pattern equivalence

equivalence = same distribution

Endhered twist

Left endhered twist
All runs of consecutive **starting points** are reversed.

Right endhered twist
All runs of consecutive **ending points** are reversed.

Example of right twist:

Thanks to an endhered twist
⌢ and ⌢ **have the same distribution!**

Goulden-Jackson cluster method and endhered patterns

Endhered pattern enumeration in matchings

Imagine we have g.f. for a distribution of a given pattern μ:

$$D_\mu(z, u) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} d_{n,k} z^n u^k.$$

There are $d_{n,k}$ matchings of size n with k occurrences of μ.

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We label some occurrences by variable v,
i.e. u is replaced either by 1 or by v.

$$H_\mu(z, v) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} h_{n,k} z^n v^k = D_\mu(z, 1 + v).$$

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It is simplier to construct H_μ than D_μ !

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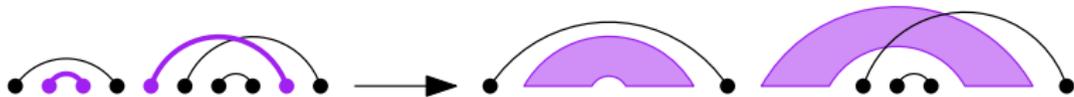i.e. u is replaced either by 1 or by v.

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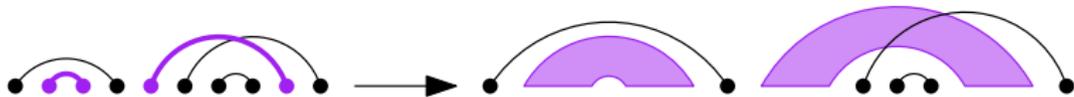**By the symbolic inclusion-exclusion:** $D_\mu(z, u) = H_\mu(z, u - 1)$

Idea: replace some arcs by patterns

(simple case without self-overlappings)

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(simple case without self-overlappings)

We label certain arcs by v, these arcs will be replaced by occurrences of pattern μ.

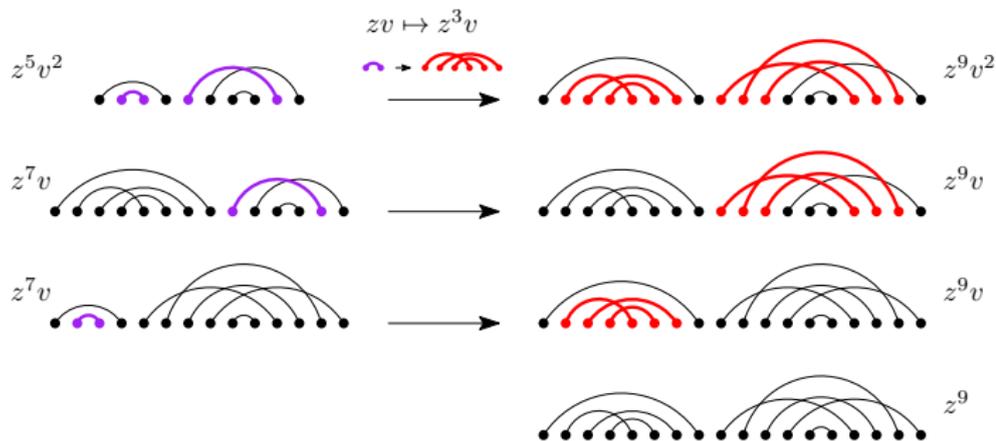$$F(z + zv),$$

where $F(z)$ is the ordinary g.f. for all matchings:

$$F(z) = \sum_{n=0}^{\infty} (2n-1)!! \, z^n = 1 + z + 3z^2 + 15z^3 + 105z^4 + \dots$$

Consider, for instance,
the endhered pattern

Zero self-overlappings

Zero self-overlappings

$zv \mapsto z^3 v$

Replace v by $u-1$
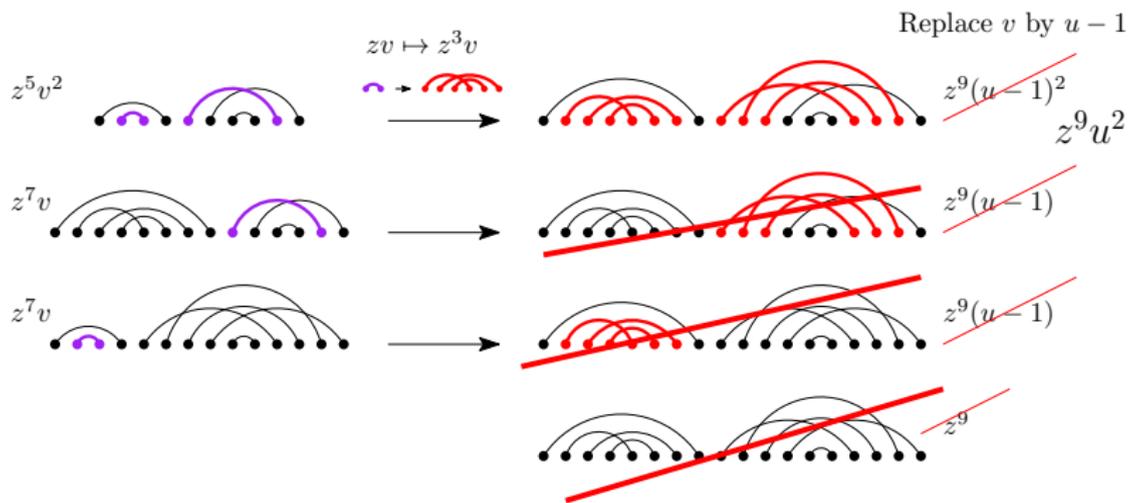
$z^5 v^2$

$z^9 (u-1)^2$

$z^7 v$

$z^9 (u-1)$

$z^7 v$

$z^9 (u-1)$

z^9
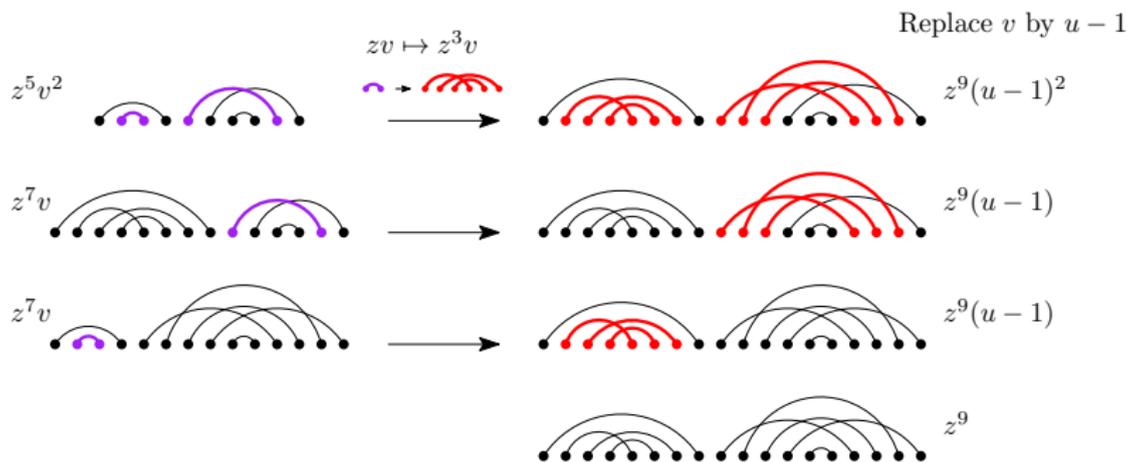
Zero self-overlappings
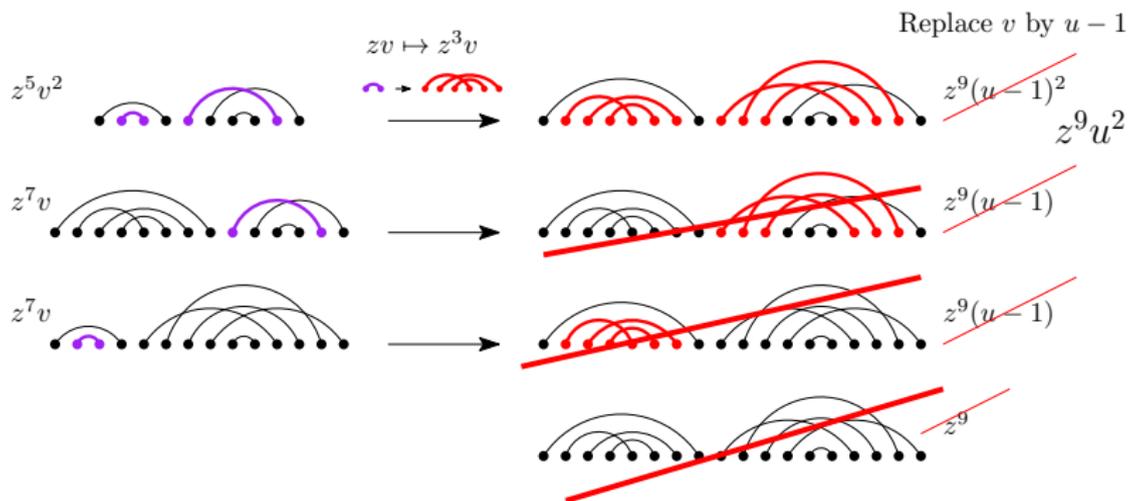
Zero self-overlappings

$$H_\mu(z, v) = F(z + z^\ell v),$$

$$D_\mu(z, u) = H_\mu(z, u - 1),$$

where ℓ is the size of pattern μ.

With
self-overlappings?

Autocorrelation encodes self-overlappings

An *autocorrelation polynomial* $A_\pi(z)$ for an endhered pattern π of size n is

$$A_\pi(z) = 1 + \sum_{k \in S} z^{n-k},$$

where S is the set of lengths of possible overlappings of two different occurrences of the pattern π in some matching.
In other words, z^{n-k} means that two occurrences have k edges in common.

Pattern	Autocorrelation
	$A_{21}(z) = 1 + z,$
	$A_{12}(z) = 1 + z,$
	$A_{132}(z) = 1,$
	$A_{321}(z) = 1 + z + z^2,$
	$A_{3412}(z) = 1 + z^2,$
	$A_{7564231}(z) = 1 + z^3 + z^6.$

Examples of endhered pattern overlappings

$vz \mapsto \frac{vz^3}{1-v(z+z^2)}$

zv → z^5v^3

zv → z^5v^2

z^2v → z^5v^2

z^2v → z^5v^2

z^3v → z^5v

z^3v → z^5v

z^3v → z^5v

z^5

$v = u - 1$ → z^5u^3

26
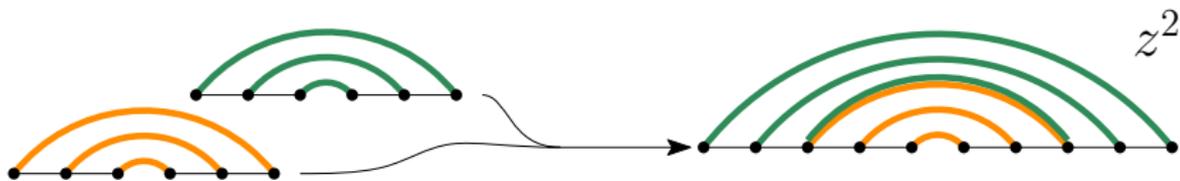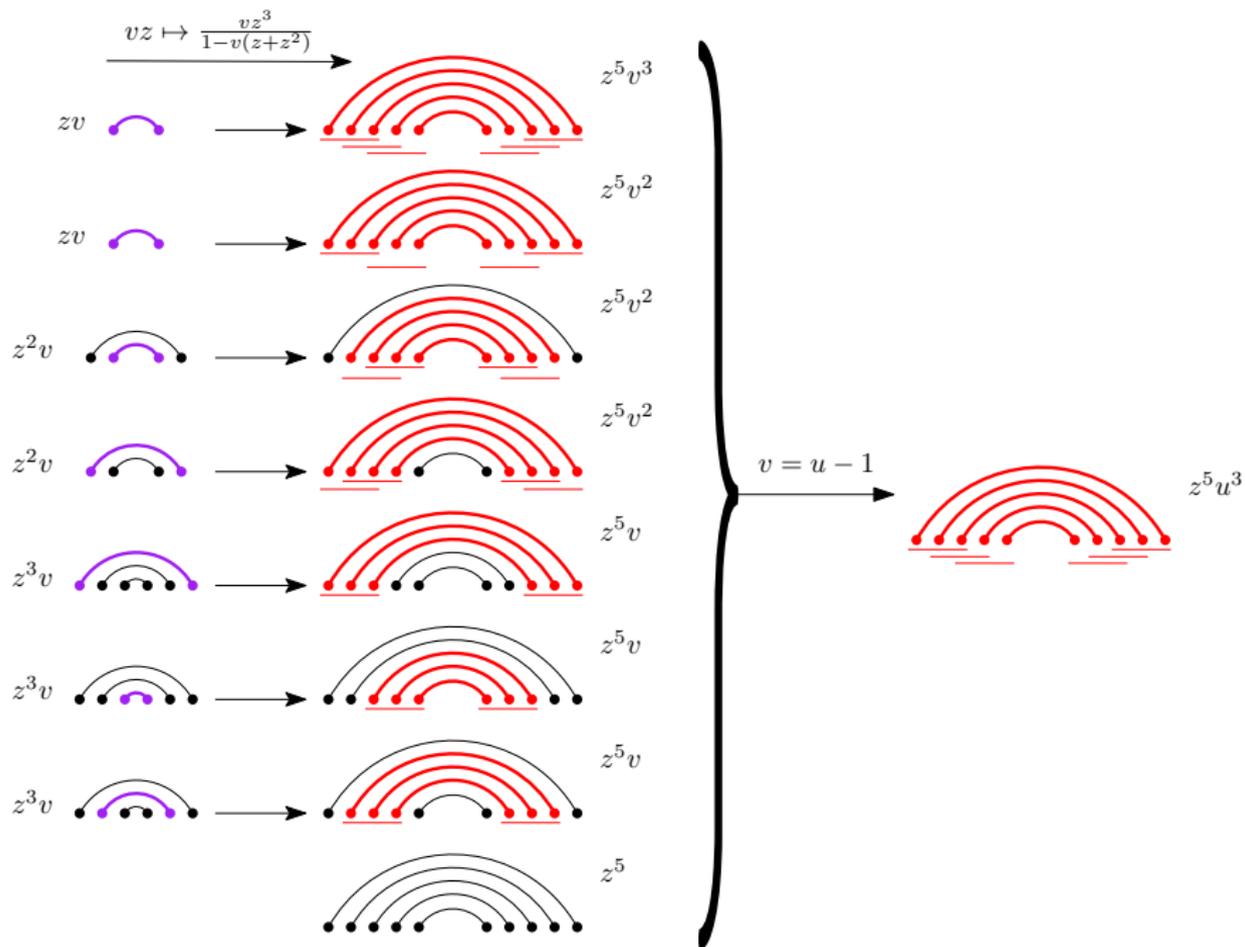
Distribution of a given pattern

Let π be an endhered pattern of size ℓ, with autocorrelation $A_\pi(z)$. The distribution is given by

$$\sum_{n,k \geq 0} a_{n,k} z^n u^k = F\left(z + \frac{(u-1)z^\ell}{1 - (u-1)(A_\pi(z) - 1)}\right),$$

where $F(z)$ is the ordinary g.f. for all matchings:

$$F(z) = \sum_{n=0}^{\infty} (2n-1)!! \, z^n = 1 + z + 3z^2 + 15z^3 + 105z^4 + \dots$$

Also possible for joint distribution of several patterns.

Well... what about the real-world data?

Snail-like pseudoknotted RNA


```
([{................)]}
```

Does it exist in nature?

Snail-like pseudoknotted RNA


```
([{................)]}
CAUAAACCGUUGGGGUCAGGUA
```
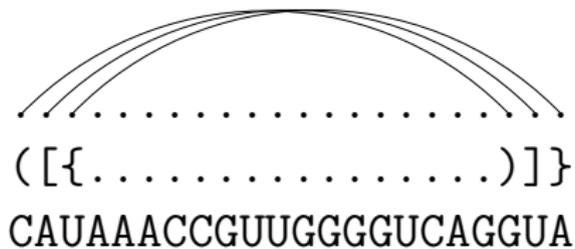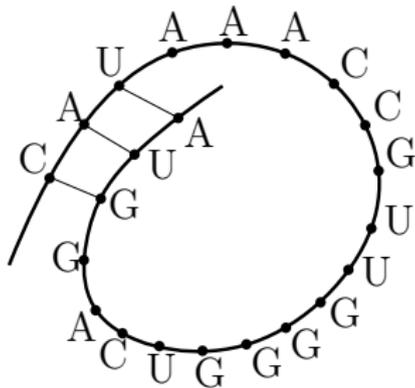
Does it exist in nature?

Interactive web application by Daniel Pinson

https://rna.kirgizov.link

Data comes from Protein Data Bank, we have used X3DNA-DSSR to obtain
matchings from 3D coordinates of atoms. FR3D Python can also be used.

Possible research directions

The distribution of endhered patterns in real-world data is different from their distribution in matchings.

> Can one characterise real-world RNA secondary structures by pattern distributions (avoidance-presence)?

> Explain possible autocorrelations or endhered patterns, (in other words, period sets)

1, 2, 4, 4, 7, 7, 11, 12, 18, ... ? (shift of A304178 ?)
Bijection with sets of palindrome prefix lengths, over all binary palindromes of length *n*???

- ✏ Endhered patterns in matchings and RNA
  Célia Biane, Greg Hampikian, Sk, Khaydar Nurligareev

- ✏ Asymptotics of self-overlapping permutations
  Sk and Khaydar Nurligareev

2 1 4 3 6 5   2 1 4 3 6 5
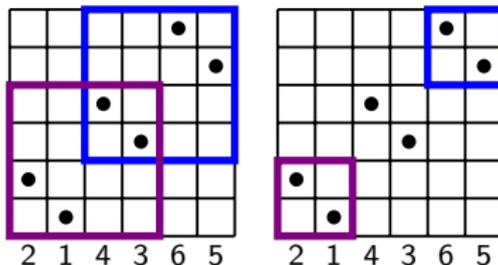
- 🖥 Interactive web application by Daniel Pinson *et al.*
  https://rna.kirgizov.link

- 💬 Distribution of endhered patterns in RNA-related secondary structures
  Célia Biane, Greg Hampikian, Sk, Khaydar Nurligareev, Daniel Pinson
  SeqBIM 2024, Rennes, 2024

- ✒ Clusters of endhered patterns in permutations and mathematics.
  In preparation. With Khaydar Nurligareev.

$\mathbb{Q}$-bonacci words

$\mathbb{Q}$-bonacci or q-decreasing words

An *n*-length binary word is *q-decreasing*, $q \in \mathbb{R}^+$, if every of its length maximal factors of the form $0^a 1^b$ satisfies $a = 0$ or $q \cdot a > b$.

$$\cdots 1 \,\Big|\, \underbrace{000\cdots\cdots 00}_{a}\underbrace{111\cdots 11}_{b} \,\Big|\, 0 \cdots$$

Let $\mathcal{W}_{q,n}$ be the set of such words of length n, $\mathcal{W}_q = \bigcup_{n \in \mathbb{N}} \mathcal{W}_{q,n}$.

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Ex.

111001010110001 is not 2-decreasing $(2 \cdot 1 \not> 2)$

01111 is not π-decreasing $(\pi \cdot 1 \not> 4)$

001111 is π-decreasing $(\pi \cdot 2 > 4)$

1-decreasing words, $\mathcal{W}_1$

In a 1-decreasing word every run of 0s is immediately
followed by a strictly shorter run of 1s.

$$\cdots 1 \;\Big|\; \underbrace{000\cdots\cdots 00}_{a}\underbrace{111\cdots 11}_{b} \;\Big|\; 0\cdots \quad a > b \text{ or } a = 0$$

Let's count!

n	1	2	3	4	$\cdots$
	2	3	5	8	Fibonacci

			0000
		000	0001
	00	001	0010
0	10	100	1000
1	11	110	1001
		111	1100
			1110
			1111

...

2-decreasing words, $\mathcal{W}_2$

$$\cdots 1 \left| \underbrace{000 \cdots\cdots 00}_{a} \underbrace{111\cdots 11}_{b} \right| 0 \cdots \quad \text{where } 2a > b \text{ or } a = 0$$

Let's count!

n	1	2	3	4	$\cdots$
	2	4	7	13	Tribonacci

			0000	
			0001	
			0010	
		000	0011	
		001	0100	
	00	010	0101	
0	01	100	1000	**...**
1	10	101	1001	
	11	110	1010	
		111	1100	
			1101	
			1110	
			1111	

34

q-decreasing words with natural *q*

> Bijections between *q*-decreasing words and words avoiding factors 1^{q+1}.

> Efficient generation and Gray codes

> Solved Eğecioğlu-Iršič conjecture
> (Hamiltonian path always exists in Fibonacci-run graphs)

📝 Gray codes for Fibonacci *q*-decreasing words.
Jean-Luc Baril, Sk and Vincent Vajnovszki
Theoretical Computer Science, 2022, https://arxiv.org/abs/2010.09505

📝 Fibonacci-run graphs I: Basic properties. Ömer Eğecioğlu and Vesna Iršič
Discrete Applied Mathematics, 2021, https://arxiv.org/abs/2010.05518

📝 Asymptotic bit frequency in Fibonacci words. BKV, GASCom 2022
https://kirgizov.link/talks/gascom2022.pdf
Pure Mathematics and Applications, 2022, https://arxiv.org/abs/2106.13550

💬 Qubonacci words. BKV
Permutations patterns 2021, https://kirgizov.link/talks/qubonacci.pdf

1-Gray code for 1-decreasing words

Eğecioğlu-Iršič conjecture is solved!

Fibonacci-run graphs for small values of *n*. Vertices correspond to the reverse of words from $\mathcal{W}_n^1$ beginning with 0.

q-decreasing words with rational and real *q*

💬 Q-bonacci words and numbers. Sk, Fibonacci conference
✒ The Fibonacci Quarterly, 2022
https://kirgizov.link/talks/fiboconf.pdf
https://arxiv.org/abs/2201.00782

💬 Sturm meets Fibonacci in Minkowski's fractal bar.
Sergey Dovgal and Sk
Permutation Patterns 2023
https://kirgizov.link/publications/pp23.pdf

✒ Structure and growth of $\mathbb{R}$-bonacci words.
Sergey Dovgal and Sk
Electronic Journal of Combinatorics, 2025
http://arxiv.org/abs/2310.01213

How fast does $|\mathcal{W}_{q,n}|$ grow ?

Consider the following function

$$\Phi(q) = \lim_{n \to \infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$$

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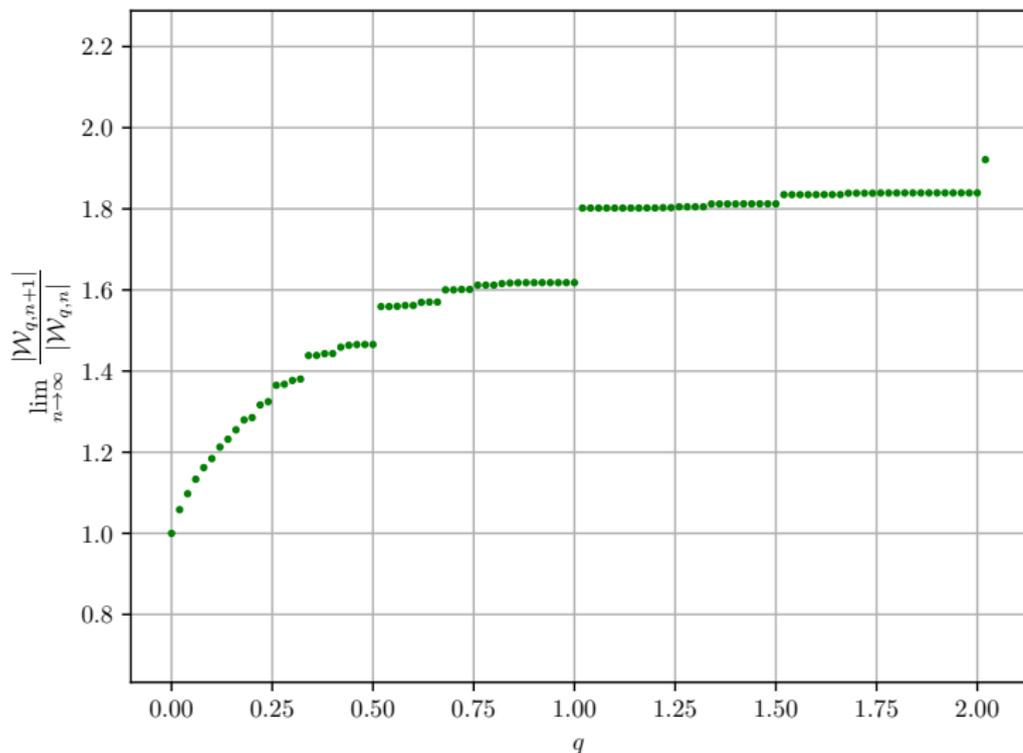$$\Phi(q) = \lim_{n \to \infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$$

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For $q = 5/3$?

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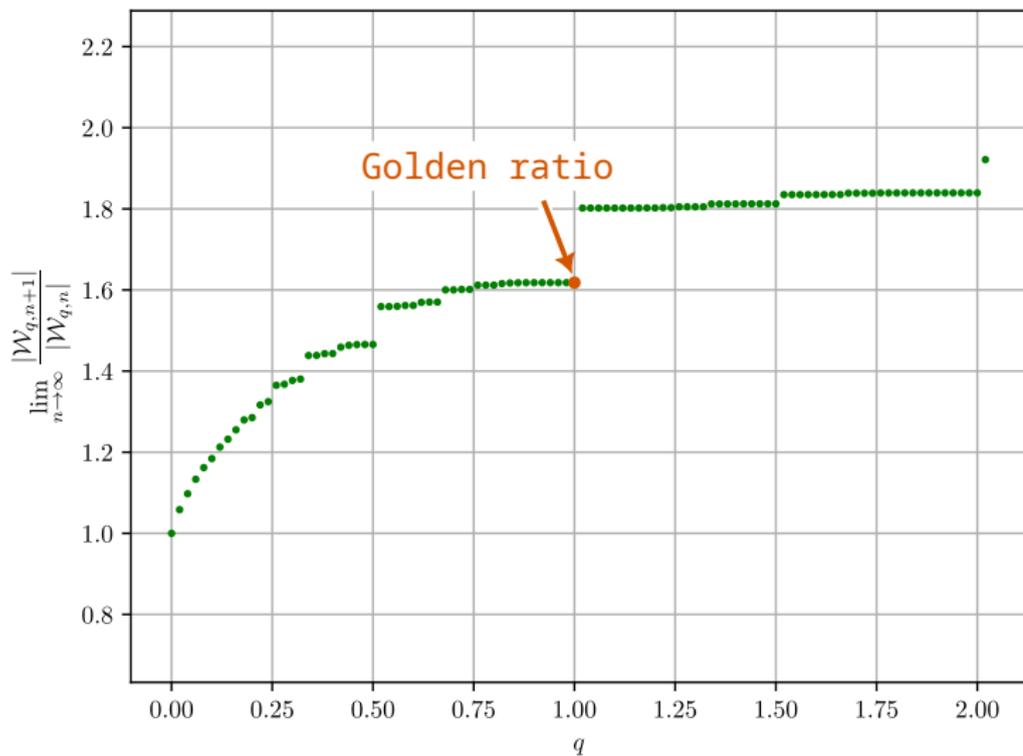For $q = 2$, it is the tribonacci constant.
For $q = 5/3$?
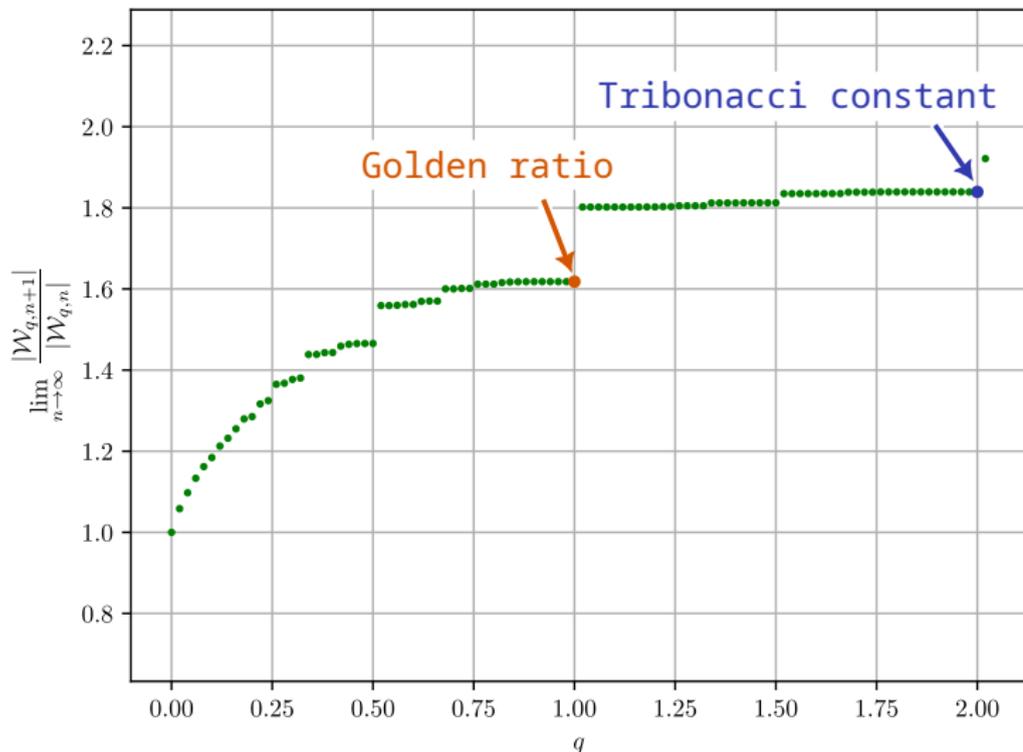For $q = \varphi$?

GeneℝAlization of the golden ratio

$$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|} \text{ as a function of } q.$$

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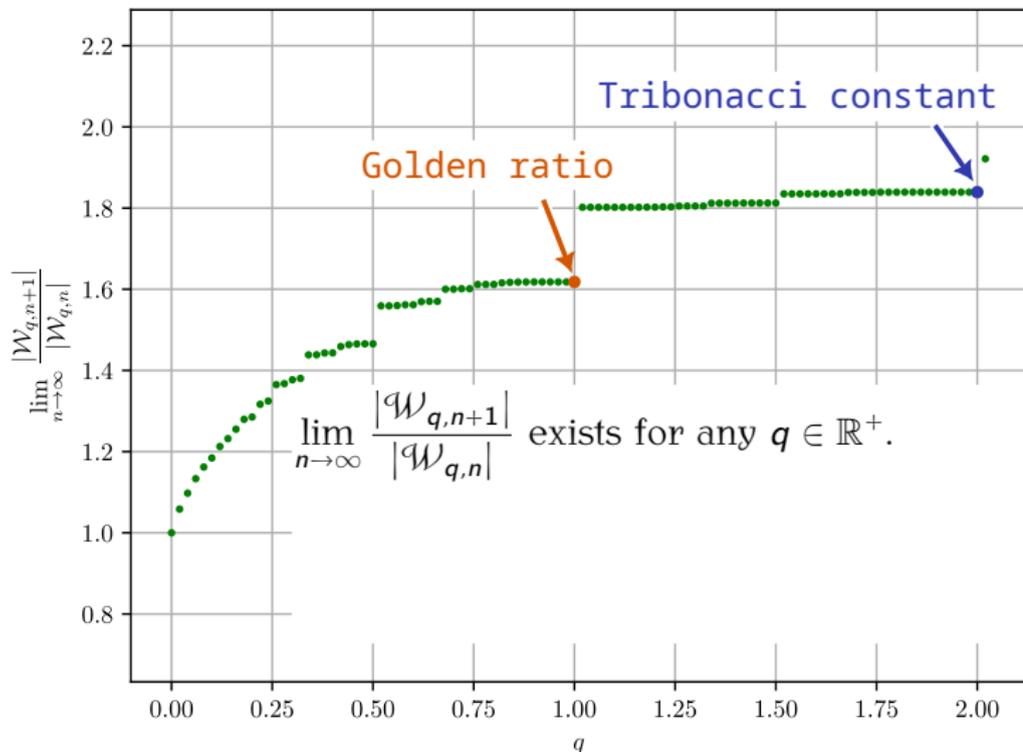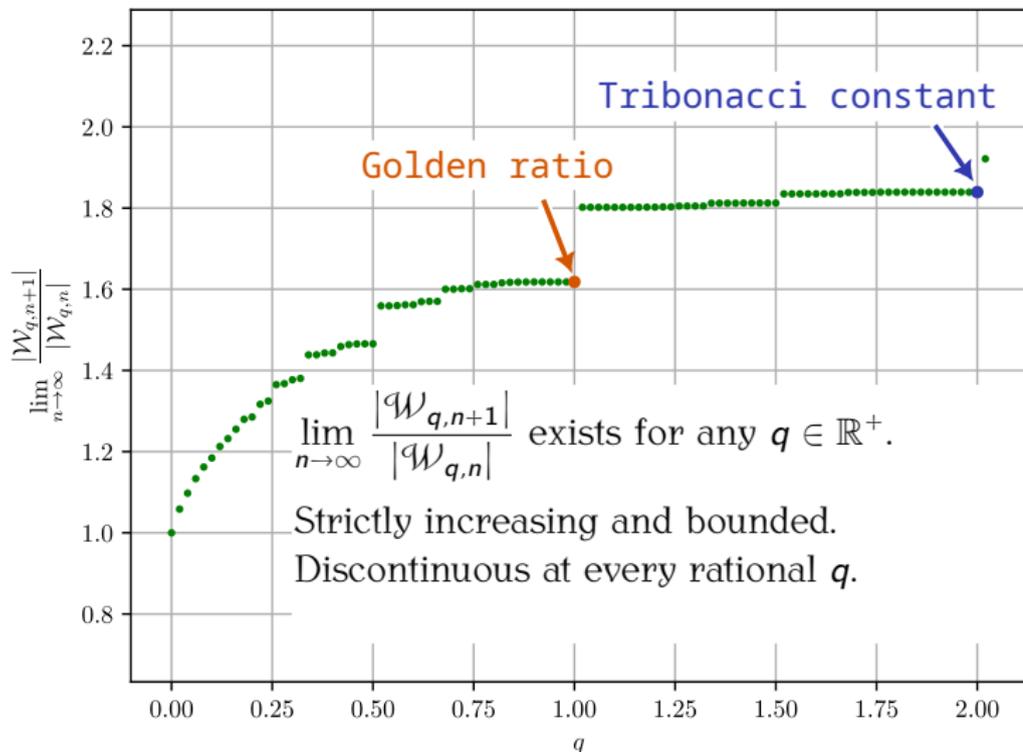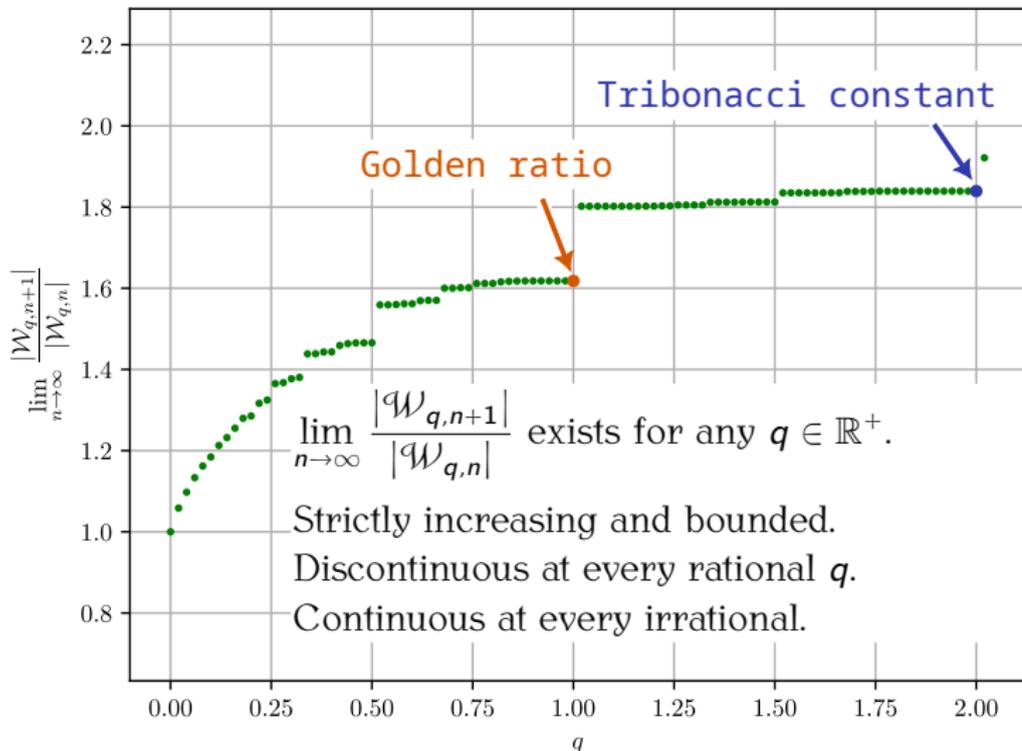# GeneℝRalization of the golden ratio

$$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|} \text{ as a function of } q.$$

Gene$\mathbb{R}$alization of the golden ratio

Tribonacci constant

Golden ratio

$\displaystyle\lim_{n\to\infty}\frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ exists for any $q \in \mathbb{R}^+$.

Strictly increasing and bounded.
Discontinuous at every rational q.

q

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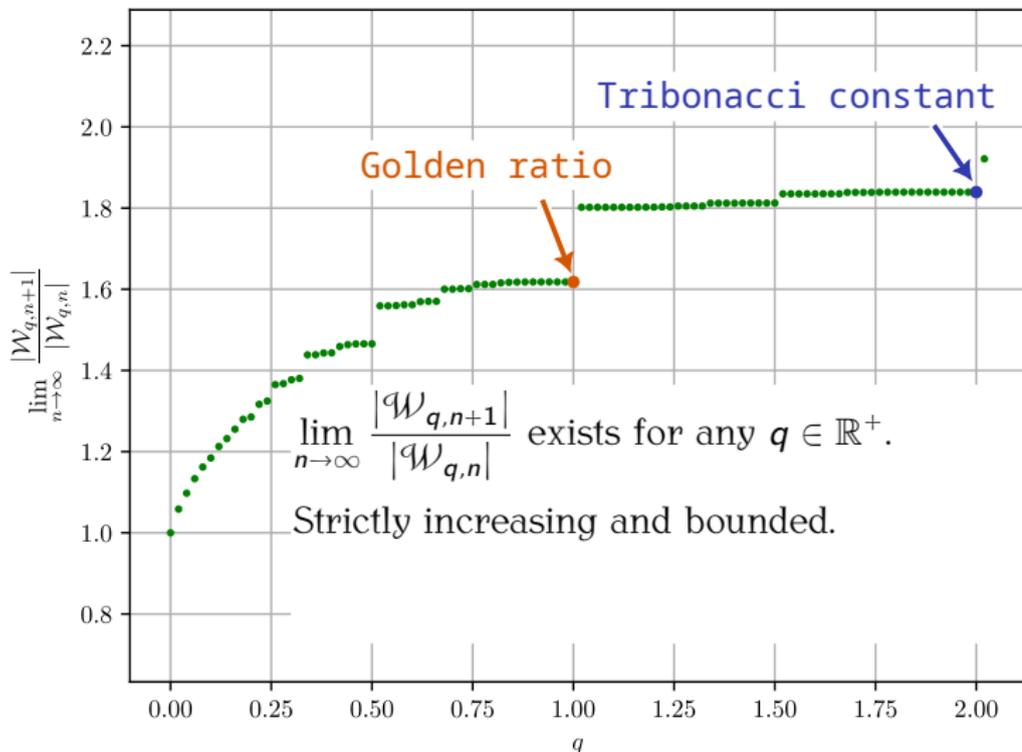
Golden ratio

$$\lim_{n \to \infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|} \text{ as a function of } q.$$

Open questions for $\mathbb{Q}$-bonacci words

In a *k-Gray code* consecutive words differs in at most *k* positions.

Conjecture. There is a 1-Gray code for *q*-decreasing binary words of size *n*, for any $q \in \mathbb{N}$.

In 2022, in the original paper, Baril, K. and Vajnovszki presented a 1-Gray code for $q = 1$ and 3-Gray code for $q > 1$.

In 2024, Wong, Liu, Lam and Im found a 2-Gray code for $q > 1$.

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For some values of *n* and $q \notin \mathbb{N}$ no 1-Gray code can exist. For example, $q = 2/3$, $n = 5$.

More!

Beautiful links with Dyck paths, integer compositions:

✎ Dyck paths enumerated by the $\mathbb{Q}$-bonacci numbers.
Elena Barcucci, Antonio Bernini, Stefano Bilotta, Renzo Pinzani
GASCom 2024, E. Proceedings in Theoretical Computer Science

✎ Rational Dyck paths.
Elena Barcucci, Antonio Bernini, Stefano Bilotta, Renzo Pinzani
Journal of Integer Sequences, 2025.

A new set of pattern avoiding binary words in bijection with
q-decreasing words, for $q \in \mathbb{Q}^+$:

✎ Pattern avoiding and q-decreasing binary words.
Elena Barcucci, Antonio Bernini, Stefano Bilotta, Renzo Pinzani
RAIRO - Theoretical Informatics and Applications, 2025

Even more!

- 📕 Fibonacci Cubes with Applications and Variations.
  Ömer Eğecioğlu, Sandi Klavžar and Michel Mollard
  World Scientific, 2023

- 📕 Combinatorial Gray codes-an updated survey, Torsten Mütze
  https://arxiv.org/pdf/2202.01280.pdf
  Electronic Journal of Combinatorics, 2022

- ✏️ Generating cyclic 2-Gray codes for Fibonacci q-decreasing words.
  Dennis Wong, Bowie Liu, Chan-Tong Lam and Marcus Im
  WALCOM: Algorithms and Computation, 2024.

- ✏️ Greedy gray codes for some restricted classes of binary words.
  Nathanaël Hassler, V. Vajnovszki and Dennis Wong
  GASCom 2024, E. Proceedings in Theoretical Computer Science

- ✏️ Enumeration in the lattice of q-decreasing words.
  Jean-Luc Baril, Nathanaël Hassler and Sk.
  Preprint. https://arxiv.org/abs/2511.09480

Perspectives

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> Nathanaël Hassler will defend his PhD this year.
> Invited Speaker at GASCom 2026, Italy, June 8-12 2026

> Finish and submit papers currently in preparation.

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> Finish and submit papers currently in preparation.

> Investigate pattern avoidance in words produced by a given grammar.
> Try to obtain an European Research Council grant.
> Continue my dilettante philosophical explorations.
> More cooperation with life sciences.

Merci !

Knuth-Fibonacci words

Poetry metrics

da DUM   da DUM   da DUM …

DUM da   DUM da   DUM da …

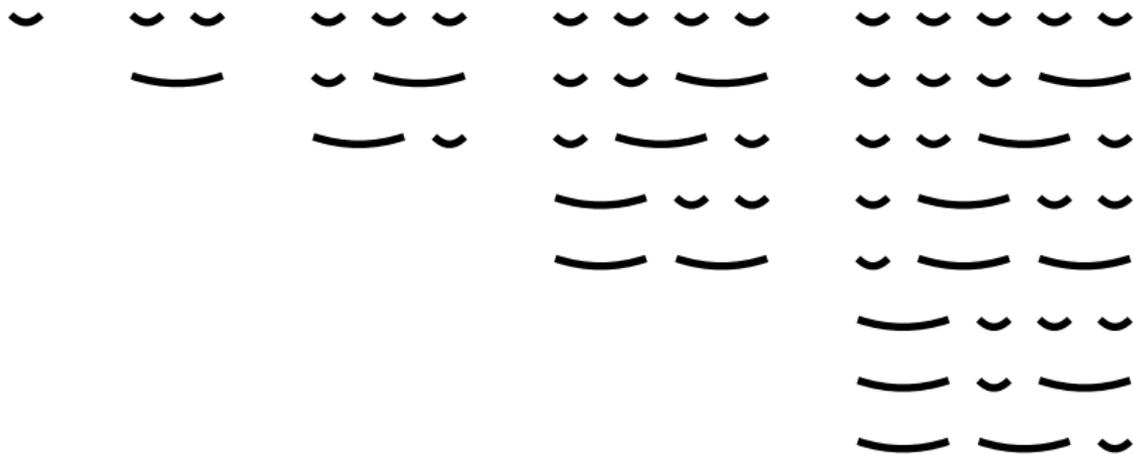da da DUM   da da DUM …

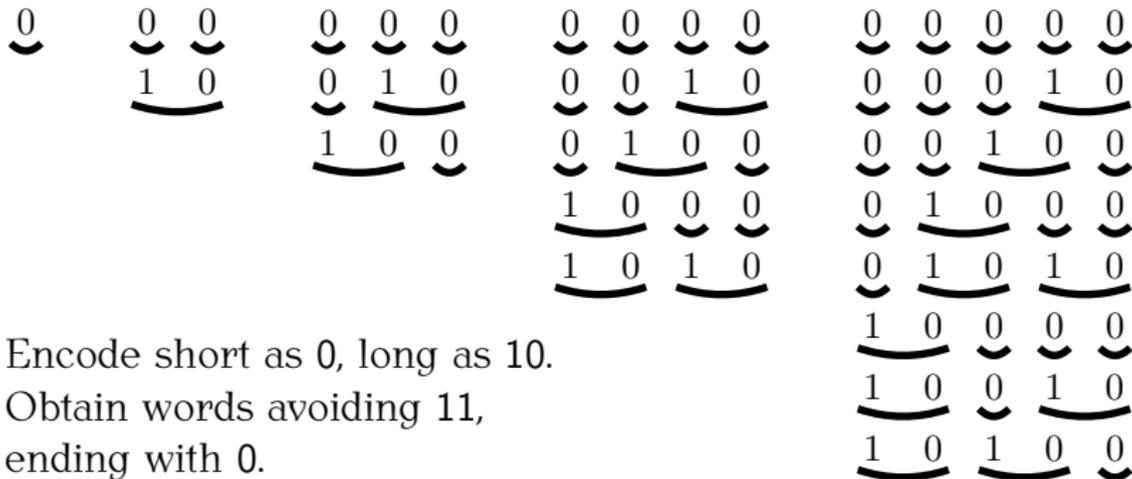da DUM da   da DUM da   da DUM da …

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In Ancient Greek, Latin, Sanskrit and some other languages there are two types of syllables: short and long.

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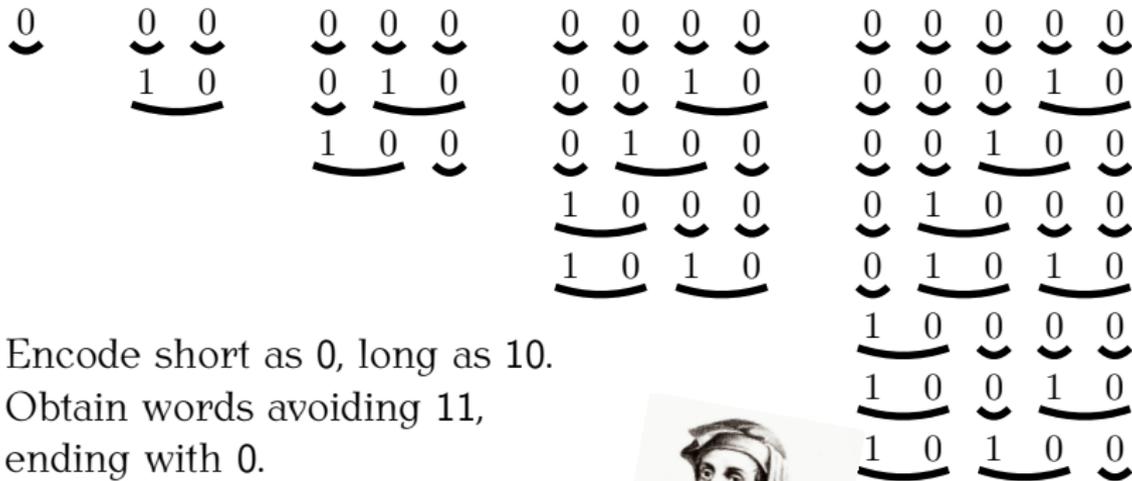

Encode short as 0, long as 10.
Obtain words avoiding 11,
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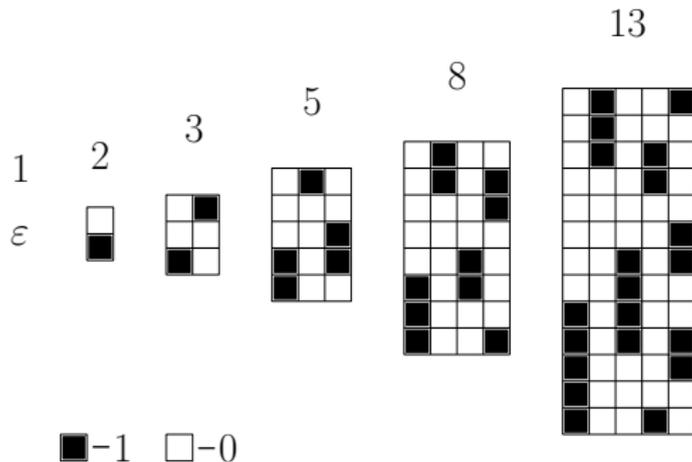1, 2, 3, 5, 8,...

Knuth-Fibonacci words

Words avoiding 1^k are counted by generalized k-step Fibonacci numbers.

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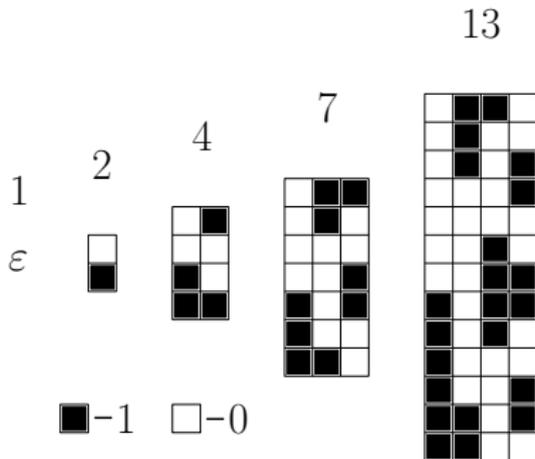Words avoiding 11 are counted by Fibonacci

$$a_n = a_{n-1} + a_{n-2}$$

Words are listed in Gray order (consecutive differs in only 1 bit)

Knuth-Fibonacci words

Words avoiding 1^k are counted by generalized k-step Fibonacci numbers.

Words avoiding 111 are counted by Tribonacci

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

Words are listed in Gray order (consecutive differs in only 1 bit)

Knuth-Fibonacci words literature

- 📕 The Art of Computer Programming, Vol. 3: Sorting, and Searching, 2 ed. (page 286), 1998, Donald Knuth

- 📕 Matters Computational (Section 14.2), 2010, Jörg Arndt
  https://www.jjj.de/fxt/fxtbook.pdf

- ✏️ Generalized Fibonacci cubes are mostly Hamiltonian
  Jenshiuh Liu, Wen-Jing Hsu, Moon Jung Chung, 1994

- ✏️ Gray codes for A-free strings. Matthew B. Squire, 1996

- ✏️ A loopless generation of bitstrings without p consecutive ones
  Vincent Vajnovszki, 2001

- ✏️ An O(1) time algorithm for generating Fibonacci strings
  Kenji Mikawa and Ishiro Semba, 2005