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Transformation à la Foata for Special Kinds of Descents and Excedances

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Abstract: A pure excedance in a permutation $\pi = \pi_1\pi_2\dots\pi_n$ is a position $i < \pi_i$ such that there is no $j < i$ with $i \leq \pi_j < \pi_i$. We present a one-to-one correspondence on the symmetric group that transports pure excedances to descents of a special kind. As a byproduct, we prove that the popularity of pure excedances equals those of pure descents on permutations, while their distributions are different.

Keywords: Cycle; Descent; Distribution; Excedance; Permutation; Popularity; Statistic
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1. Introduction and notations

The distribution of the number of descents has been widely studied on several classes of combinatorial objects such as permutations [14], cycles [7, 8], and words [3, 10]. Many interpretations of this statistic appear in several fields as Coxeter groups [4, 11] or lattice path theory [12]. One of the most famous result involves the *Foata fundamental transformation* [9] to establish a one-to-one correspondence between descents and excedances on permutations. This bijection provides a more straightforward proof than those of MacMahon [14] for the equidistribution of these two Eulerian statistics.

In this paper, we present a bijection *à la Foata* on the symmetric group that exchanges pure excedances with special kind of descents defined as a mesh pattern p_2 [6] (see below for the definition of this pattern). Then, we deduce that the popularities (but not the distributions) of pure descents [2] and pure excedances are the same. This common popularity is given by the generalized Stirling number $n! \cdot (H_n - 1)$ (see Sequence A001705 in [15]) where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number. Finally, we conjecture the existence of a bijection on the symmetric group that exchanges pure excedances and p_2 while preserving the number of cycles.

Let S_n be the set of permutations of length n, *i.e.*, all bijections from $[n] = \{1, 2, \dots, n\}$ into itself. The one-line representation of a permutation $\pi \in S_n$ is $\pi = \pi_1\pi_2\dots\pi_n$ where $\pi_i = \pi(i)$, $1 \leq i \leq n$. For $\sigma \in S_n$, the *product* $\sigma \cdot \pi$ is the permutation $\sigma(\pi_1)\sigma(\pi_2)\dots\sigma(\pi_n)$. A ℓ*-cycle* $\pi = \langle i_1, i_2, \dots, i_\ell\rangle$ in S_n is a n-length permutation satisfying $\pi(i_1) = i_2, \pi(i_2) = i_3, \dots, \pi(i_{\ell-1}) = i_\ell, \pi(i_\ell) = i_1$ and $\pi(j) = j$ for $j \in [n]\backslash\{i_1, i_2, \dots, i_\ell\}$. For $1 \leq k \leq n$, we denote by $C_{n,k}$ the set of all n-length permutations admitting a decomposition in a product of k disjoint cycles. The set $C_{n,k}$ is counted by the signless Stirling numbers of the first kind $c(n, k)$ defined by

$$c(n, k) = (n - 1)\, c(n - 1, k) + c(n - 1, k - 1)$$

where $c(n, k) = 0$ if $n = 0$ or $k = 0$, except $c(0, 0) = 1$ (see [16, 17] and Sequence A132393 in [15]). These numbers also enumerate n-length permutations π having k *left-to-right maxima*, *i.e.*, positions $i \in [n]$ such that $\pi_j < \pi_i$ for $j < i$ (see [16]), and permutations $\pi \in S_n$ with $k - 1$ *pure descents*, *i.e.*, descents $\pi_i > \pi_{i+1}$ where there is no $j < i$ such that $\pi_j \in [\pi_{i+1}, \pi_i]$ (see [2]). Note that a pure descent can be viewed as an occurrence of the mesh pattern $(21, L_1)$ where $L_1 = \{1\} \times [0, 2] \cup \{(0, 1)\}$. Indeed, for a k-length permutation σ and a subset $R \subseteq [0, k] \times [0, k]$, an occurrence of the mesh pattern (σ, R) in a permutation π is an occurrence of σ in π with the additional restriction that no element of π lies inside the shaded regions defined by R, where $(i, j) \in R$ means the square having bottom left corner (i, j) in the graphical representation $\{(i, \sigma_i), i \in [k]\}$ of σ. For instance, an occurrence of the mesh pattern p_1 in Figure 1 corresponds to an occurrence of a pure descent. See [6] for a more detailed definition of mesh patterns.

Regarding this interpretation of pure descents in terms of mesh patterns, we define other kinds of descents by the mesh patterns $p_i = (21, L_i)$, $p'_i = (21, R_i)$ with $L_i = \{1\} \times [0, 2] \cup \{(0, i)\}$ and $R_i = \{1\} \times [0, 2] \cup \{(2, i)\}$ for $0 \leq i \leq 2$. Modulo the trivial symmetries on permutations (reverse and complement), it is straightforward to see that p_0, p_1 and p_2 are respectively in the same distribution class as p'_2, p'_1 and p'_0. Then, we deal with only mesh patterns p_i, $i \in [0, 2]$. We refer to Figure 1 for a graphical illustration. On the other hand, we define a *pure excedance* as an occurrence of an excedance, *i.e.* $\pi_i > i$, with the additional restriction that there is no point (j, π_j) such that $1 \le j \le i-1$ with $i \le \pi_j < \pi_i$. Although such a pattern (called *pex*) is not a mesh pattern, we can represent it graphically as shown in Figure 1.

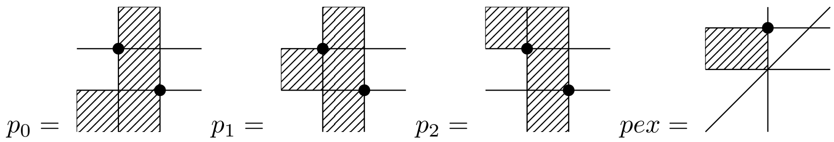


Figure 1: Illustration of the mesh patterns p_0, p_1, p_2 and pex; p_1 and pex correspond respectively to a pure descent and a pure excedance.

A *statistic* is an integer-valued function from a set $\mathcal{A}$ of n-length permutations (we use the boldface to denote statistics). For a pattern p, we define the pattern statistic $\mathbf{p} : \mathcal{A} \to \mathbb{N}$ where the image $\mathbf{p}\ \pi$ of $\pi \in \mathcal{A}$ by $\mathbf{p}$ is the number of occurrences of p in π. The *popularity* of p in $\mathcal{A}$ is the total number of occurrences of p over all objects of $\mathcal{A}$, that is $\sum_{a \in \mathcal{A}} \mathbf{p}\ a$ (see [5] for instance). Below, we present statistics that we use throughout the paper:

$$
\begin{aligned}
\mathbf{exc}\ \pi &= \text{number of excedances in } \pi,\\
\mathbf{pex}\ \pi &= \text{number of pure excedances in } \pi,\\
\mathbf{des}\ \pi &= \text{number of descents in } \pi,\\
\mathbf{des}_i\ \pi &= \text{number of patterns } p_i \text{ in } \pi,\ 0 \le i \le 2,\\
\mathbf{fix}\ \pi &= \text{number of fixed points in } \pi,\\
\mathbf{cyc}\ \pi &= \text{number of cycles in the decomposition of } \pi,\\
\mathbf{pcyc}\ \pi &= \text{number of pure cycles (i.e. cycles of length at least two) in } \pi,\\
&= \mathbf{cyc}\ \pi - \mathbf{fix}\ \pi
\end{aligned}
$$

We organize the paper as follows. In Section 2, we focus on patterns p_i, $0 \le i \le 2$. We prove that the statistics $\mathbf{des}_0$ and $\mathbf{des}_1$ are equidistributed by giving algebraic and bijective proofs. Next, we provide the bivariate exponential generating function for the distribution of p_2, and we deduce that p_2 has the same popularity as p_0 and p_1, without having the same distribution. In Section 3, we present a bijection on S_n that transports pure excedances into patterns p_2. Notice that the Foata's first transformation [9] is not a candidate for such a bijection. As a consequence, pure descents and pure excedances are equipopular on S_n, but they do not have the same distribution. Combining all these results, we deduce that patterns p_i, $0 \le i \le 2$, and *pex* are equipopular on the symmetric group S_n. Finally we present two conjectures about the equidistribution of $(\mathbf{cyc}, \mathbf{des}_2)$ and $(\mathbf{cyc}, \mathbf{pex})$, and that of $(\mathbf{des}, \mathbf{des}_2)$ and $(\mathbf{exc}, \mathbf{pex})$.

2. The statistics $\mathbf{des}_i$, $0 \le i \le 2$

For $0 \le i \le 2$, let $A^i_{n,k}$ be the set of n-length permutations having k occurrences of p_i, and denote by $a^i_{n,k}$ its cardinality. Let $A^i(x,y)$ be the bivariate exponential generating function $\sum_{n=0}^{\infty} \sum_{k=0}^{n-1} a^i_{n,k} \frac{x^n}{n!} y^k$. In [2, 13], it is proved that $a^1_{n,k}$ equals the signless Stirling numbers of the first kind $c(n, k+1)$ (see Sequence A132393 in [15]). Indeed, a permutation $\sigma \in A^1_{n,k}$ can be uniquely obtained from an $(n-1)$-length permutation π by one of the two following constructions:

(i) if $\pi \in A^1_{n-1,k-1}$, then we increase by one all values of π greater than or equal to π_{n-1}, and we add π_{n-1} at the end;

(ii) if $\pi \in A^1_{n-1,k}$, then we increase by one all values of π greater than or equal to a given value $x \le n$, $x \ne \pi_{n-1}$ and we add x at the end.

Then, we deduce the recurrence relation $a^1_{n,k} = a^1_{n-1,k-1} + (n-1)a^1_{n-1,k}$ with $a^1_{n,0} = (n-1)!$ for $n \ge 1$, $a^1_{0,0} = 1$ and the bivariate exponential generating function is

$$A^1(x,y) = \frac{1}{y(1-x)^y} - \frac{1}{y} + 1$$

which proves that $a^1_{n,k} = c(n, k+1)$.

Below, we prove that $a^1_{n,k}$ also counts n-length permutations having k occurrences of the pattern p_0.

Theorem 2.1. *The number $a^0_{n,k}$ of n-length permutations having k occurrences of pattern p_0 equals $a^1_{n,k} = c(n, k+1)$.*

Proof. An n-length permutation $\sigma \in A^0_{n,k}$ can be uniquely obtained from an $(n-1)$-length permutation π by one of the two following constructions:

(i) if $\pi \in A^0_{n-1,k-1}$, then we increase by one all values of π and we add 1 at the end;

(ii) if $\pi \in A^0_{n-1,k}$, then we increase by one all values of π greater than or equal to a given value x, $1 < x \leq n$, and we add x at the end.

We deduce the recurrence relation $a^0_{n,k} = a^0_{n-1,k-1} + (n-1)a^0_{n-1,k}$ with the initial condition $a^0_{n,0} = (n-1)!$, and then $a^0_{n,k} = a^1_{n,k} = c(n, k+1)$. □

Now, we focus on the distribution of the pattern p_2. Table 1 provides exact values for small sizes.

Theorem 2.2. *We have*

$$A^2(x, y) = \frac{e^{x(1-y)}}{(1-x)^y},$$

and the general term $a^2_{n,k}$ satisfies for $n \geq 2$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$

$$a^2_{n,k} = na^2_{n-1,k} + (n-1)a^2_{n-2,k-1} - (n-1)a^2_{n-2,k}$$

with the initial conditions $a^2_{n,0} = 1$ and $a^2_{n,k} = 0$ for $n \geq 0$ and $k > \lfloor \frac{n}{2} \rfloor$ (see Table 1 and Sequence A136394 in [15]).

Proof. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ denote a permutation of length n having k occurrences of pattern p_2. Let $u_{n,k}$ (resp. $v_{n,k}$) be the number of such permutations satisfying $\sigma_n = n$ (resp. $\sigma_n < n$). Obviously, we have

$$a^2_{n,k} = u_{n,k} + v_{n,k}.$$

A permutation σ with $\sigma_n = n$ can be uniquely constructed from an $(n-1)$-length permutation π as $\sigma = \pi_1\pi_2 \ldots \pi_{n-1}n$. No new occurrences of p_2 are created, and we obtain

$$u_{n,k} = a^2_{n-1,k}.$$

A permutation σ satisfying $\sigma_n < n$ can be uniquely obtained from an $(n-1)$-length permutation π by adding a value $x < n$ on the right side of its one-line notation, after increasing by one all the values greater than or equal to x. This construction creates a new pattern p_2 if and only if π ends with $n-1$. Thus, we deduce

$$v_{n,k} = (n-1)u_{n-1,k-1} + (n-1)v_{n-1,k}.$$

Combining the equations, we obtain for $n \geq 2$ and $k \geq 1$

$$a^2_{n,k} = na^2_{n-1,k} + (n-1)a^2_{n-2,k-1} - (n-1)a^2_{n-2,k},$$

which implies the following differential equation

$$\frac{\partial A^2(x,y)}{\partial x} = (y-1)xA^2(x,y) + \frac{\partial \left(xA^2(x,y)\right)}{\partial x}, \text{ where } A^2(x,0) = 1.$$

A simple calculation provides the claimed closed form for the generating function $A^2(x, y)$. □

Corollary 2.1. *For $0 \leq i \leq 2$, the patterns p_i are equipopular on S_n. Their popularity is given by the generalized Stirling number $n! \cdot (H_n - 1)$ (see Sequence A001705 in [15]) where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number.*

Proof. The generating function of the popularity is directly deduced from the bivariate generating function of pattern distribution by calculating

$$\left.\frac{\partial A^1(x,y)}{\partial y}\right|_{y=1} = \left.\frac{\partial A^2(x,y)}{\partial y}\right|_{y=1}.$$

□

The statistic $\mathbf{des}_2$ has a different distribution from $\mathbf{des}_0$ and $\mathbf{des}_1$, but the three patterns p_0, p_1, p_2 have the same popularity. Below we present a bijection on S_n that transports the statistic $\mathbf{des}_2$ to the statistics $\mathbf{pcyc} = \mathbf{cyc} - \mathbf{fix}$.

$k\backslash n$	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1
1		1	5	20	84	409	2365	16064
2				3	35	295	2359	19670
3						15	315	4480
4								105
...								...
$\sum$	1	2	6	24	120	720	5040	40320

Table 1: Number of n-length permutations having k occurrences of p_2 for $0 \leq k \leq 4$ and $1 \leq n \leq 8$.

Theorem 2.3. *There is a one-to-one correspondence ϕ on S_n such that for any $\pi \in S_n$, we have*

$$\mathbf{des}_2\ \pi = \mathbf{pcyc}\ \phi(\pi).$$

Proof. Let π be a permutation of length n having k occurrences of p_2. We decompose

$$\pi = B_0\pi_{i_1}A_1B_1\pi_{i_2}A_2B_2\pi_{i_3}\ldots\pi_{i_k}A_kB_k,$$

where
- $\pi_{i_1} < \pi_{i_2} < \ldots < \pi_{i_k}$ are the tops of the occurrences of p_2, *i.e.* values $\pi_{i_j} > \pi_{i_j+1}$ such that there does not exist $\ell < i_j$ such that $\pi_\ell > \pi_{i_j}$,
- A_j is a maximal sequence such that all its values are lower than π_{i_j},
- for $0 \leq j \leq k$, B_j is an increasing sequence such that $\pi_{i_j} < \min B_j$ and $\max B_j < \pi_{i_{j+1}}$.

Now we construct an n-length permutation $\phi(\pi)$ with k pure cycles as follows:

$$\phi(\pi) = \langle \pi_{i_1}A_1\rangle \cdot \langle \pi_{i_2}A_2\rangle \cdots \langle \pi_{i_k}A_k\rangle.$$

For instance, if $\pi = 125346879$ then $\phi(\pi) = \langle 5, 3, 4\rangle \cdot \langle 8, 7\rangle$. The map ϕ is clearly a bijection on S_n such that $\mathbf{des}_2\ \pi$ equals the number of pure cycles in $\phi(\pi)$. $\square$

Note that ϕ^{-1} is closely related to the Foata fundamental transformation [9].

3. The statistic pex of pure excedances

In order to prove the equidistribution of **pex** and $\mathbf{des}_2$, regarding Theorem 2.3, it suffices to construct a bijection on S_n that transports pure excedances to pure cycles. Here, we first exhibit a bijection on the set D_n of n-length derangements (permutations without fixed points), then we extend it to the set of all permutations S_n.

Any permutation $\pi \in S_n$ is uniquely decomposed as a product of transpositions of the following form:

$$\pi = \langle t_1, 1\rangle \cdot \langle t_2, 2\rangle \cdots \langle t_n, n\rangle$$

where t_i are integers such that $1 \leq t_i \leq i$. The transposition array of π is defined by $T(\pi) = t_1t_2\ldots t_n$, which induces a bijection $\pi \longmapsto T(\pi)$ from S_n to the product set $T_n = [1] \times [2] \times \cdots \times [n]$. By Lemma 1 from [1], the number of cycles of a permutation π is given by the number of fixed points in $T(\pi)$. Moreover, it is straightforward to check the two following properties:
- if $t_i = i$, then $\pi_i = i$ if and only if there is no number $j > i$ such that $t_j = t_i = i$;
- if $t_i = i$ and $\pi_i \neq i$, then i is the minimal element of a cycle of length at least two in π.

So, we deduce the following lemma.

Lemma 3.1. *The transposition array $t_1t_2\ldots t_n \in T_n$ corresponds to a derangement if and only if: $t_i = i \Rightarrow$ there is a $j > i$ such that $t_j = i$.*

Given a derangement $\pi = \pi_1\pi_2\ldots\pi_n \in D_n$ and its graphical representation $\{(i, \pi_i), i \in [n]\}$. We say that the square $(i, j) \in [n] \times [n]$ is *free* if all following conditions hold:

(i) Neither π_i nor i is a position of a pure excedance;

(ii) (i, j) is not on the first diagonal, *i.e.* $j \neq i$;

(iii) there does not exist $k > i$ such that $\pi_k = j$;

(iv) j is not a pure excedance such that $j < i$ and $\pi^{-1}(j) < i$;

(v) there does not exist $k < i$, with $\pi_k = j > i$ such that all values of the interval $[i, j-1]$ appear on the right of π_i in π.

Whenever at least one of the statements above is not satisfied, we say that the square (i, j) is *unfree*. Notice that if i and π_i are not the positions of a pure excedance, then the square (i, π_i) is always free. So, for a column i of the graphical representation of π such that i and π_i are not the positions of a pure excedance, we label by j the jth free square from the bottom to the top. We refer to Figure 2 for an example of this labeling.

Now we define the map λ from D_n to the set $T_n^\bullet$ of transposition arrays of length n satisfying the property of Lemma 3.1.

For a permutation $\pi = \pi_1\pi_2\ldots\pi_n \in D_n$, we label its graphical representation as defined above, and $\lambda(\pi) = \lambda_1\lambda_2\ldots\lambda_n$ is obtained as follows:

- if i is a pure excedance in π, then we set $\lambda_i = i$ and $\lambda_{\pi^{-1}(i)} = i$;
- otherwise, λ_i is the sum of the label of the free square (i, π_i) with the number of pure excedances $k < i$ such that $\pi^{-1}(k) < i$.

For instance, if $\pi = 6\ 8\ 12\ 5\ 4\ 7\ 3\ 2\ 11\ 1\ 9\ 10$ then we obtain $\lambda(\pi) = 1\ 1\ 2\ 4\ 4\ 2\ 1\ 1\ 9\ 1\ 9\ 10$ (see Figure 2).

Let us consider i, $1 \le i \le n$. If i is a pure excedance of π, then we fix $\lambda_i = i$ and $\lambda_{\pi^{-1}(i)} = i < \pi^{-1}(i)$. Otherwise, the square (i, i) is unfree, and all squares (i, π_k), $i+1 \le k \le n$, are unfree, which implies that the number of free squares in the ith column is less than or equal to i. This means that $\lambda(\pi)$ lies in T_n. Note that, by construction, all labeled squares do not correspond to any pure excedance. Now let us prove that the square (i, π_i) cannot be labeled i. Indeed, if $\pi_i < i$ then the label of (i, π_i) is necessarily at most $\pi_i \le i-1$; otherwise, if $\pi_i > i$ then the fact that i is not a pure excedance implies that there is $\pi_j \in [i, \pi_i - 1]$ with $j < i$. Let us choose the lowest j with this property. Using (v), the square (i, j) is unfree, which implies that the label of (i, π_i) is less than or equal to n minus the minimal number of unfree squares (i, j) in column i, that is $n - (n - i + 1) = i - 1$. Moreover, the transposition array $\lambda(\pi)$ has exactly $\mathbf{pex}\ \pi$ fixed points, and for any fixed point i there necessarily exists $j = \pi^{-1}(i) > i$ such that $\lambda_j = \lambda_i = i$. This implies that $\lambda(\pi) \in T_n^\bullet$.

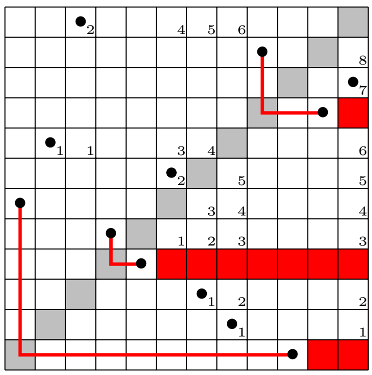
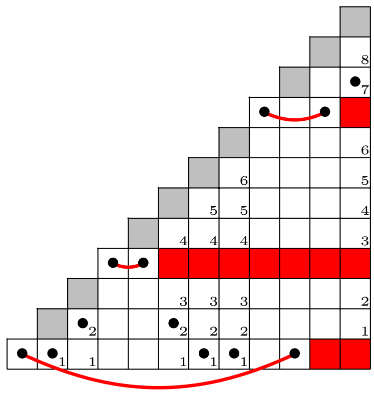
$\pi =$  $\lambda(\pi) =$ 

Figure 2: Illustration of the bijection λ for $\pi = 6\ 8\ 12\ 5\ 4\ 7\ 3\ 2\ 11\ 1\ 9\ 10$ and $\lambda(\pi) = 1\ 1\ 2\ 4\ 4\ 2\ 1\ 1\ 9\ 1\ 9\ 10$.

Theorem 3.1. *The map λ from D_n to $T_n^\bullet$ is a bijection such that*

$$\mathbf{pex}\ \pi = \mathbf{fix}\ \lambda(\pi).$$

Proof. Since the cardinality of $T_n^\bullet$ equals that of D_n, and the image of D_n by λ is contained in $T_n^\bullet$, it suffices to prove the injectivity.

Let π and σ, $\pi \neq \sigma$, be two derangements in D_n. If π and σ do not have the same pure excedances, then, by construction, $\lambda(\pi)$ and $\lambda(\sigma)$ do not have the same fixed points, and thus $\lambda(\pi) \neq \lambda(\sigma)$.

Now, let us assume that π and σ have the same pure excedances. If there is a pure excedance i such that $\pi^{-1}(i) \neq \sigma^{-1}(i)$ then the definition implies $\lambda(\pi) \neq \lambda(\sigma)$. Otherwise the two permutations have the same pure excedances i, and for each of them we have $\pi^{-1}(i) = \sigma^{-1}(i)$. Let j be the greatest integer such that $\pi_j \neq \sigma_j$ (without loss of generality, we assume $\pi_j < \sigma_j$). In this case, j is not a pure excedance for the two permutations. Thus, $\lambda(\pi)_j$ (resp. $\lambda(\sigma)_j$) is the sum of the label of (j, π_j) (resp. (j, σ_j)) with the number of pure excedances

$k < j$ such that $\pi^{-1}(k) < j$ (resp. $\sigma^{-1}(k) < j$). Since we have $\pi_j < \sigma_j$, the label of (j, π_j) is less than the label of (j, σ_j), and the number of pure excedances $k < j$ such that $\pi^{-1}(k) < j$ is less than or equal to the number of pure excedances $k < j$ such that $\sigma^{-1}(k) < j$. Then we have $\lambda(\pi)_j < \lambda(\sigma)_j$. Then λ is an injective map, and thus a bijection. ☐

Theorem 3.2. *There is a one-to-one correspondence ψ on the set D_n of n-length derangements such that for any $\pi \in D_n$,*

$$\mathbf{pex}\ \pi = \mathbf{cyc}\ \psi(\pi).$$

Proof. Considering Theorem 2.3 and Theorem 3.1, we define for any $\pi \in D_n$, $\psi(\pi) = \phi(\sigma)$ where σ is the permutation having $\lambda(\pi)$ as transposition array. ☐

Theorem 3.3. *The two bistatistics $(\mathbf{pex}, \mathbf{fix})$ and $(\mathbf{pcyc}, \mathbf{fix})$ are equidistribiuted on S_n.*

Proof. Considering Theorem 3.2, we define the map $\bar{\psi}$ on S_n. Let π' be the permutation obtained from π by deleting all fixed points and after rescaling as a permutation. Let $I = \{i_1, i_2, \ldots, i_k\}$ be the set of fixed points of π. Then, we set $\pi'' = \psi(\pi')$. So, $\sigma = \bar{\psi}(\pi)$ is obtained from π'' by inserting fixed points $i \in I$ after a shift of all other entries in order to produce a permutation in S_n. By construction, we have $\mathbf{pex}\ \pi = \mathbf{pcyc}\ \sigma$ and $\mathbf{fix}\ \pi = \mathbf{fix}\ \sigma$ which completes the proof. ☐

A byproduct of this theorem is

Corollary 3.1. *The statistics $\mathbf{cyc}$ and $\mathbf{pex} + \mathbf{fix}$ are equidistributed on S_n.*

Also, a direct consequence of Theorems 2.3 and 3.3 is

Theorem 3.4. *The two statistics $\mathbf{pex}$ and $\mathbf{des}_2$ are equidistributed on S_n.*

Notice that Foata's first transformation is not a candidate for proving the equidistribution of $\mathbf{pex}$ and $\mathbf{des}_2$, while it transports $\mathbf{exc}$ to $\mathbf{des}$. Combining Theorem 3.4 and Corollary 2.1 we have the following.

Corollary 3.2. *For $0 \leq i \leq 2$, the patterns p_i and pex are equipopular on S_n (see Sequence A001705 in [15]).*

Finally, we present two conjectures for future works.

Conjecture 3.1. *The two bistatistics $(\mathbf{des}_2, \mathbf{cyc})$ and $(\mathbf{pex}, \mathbf{cyc})$ are equidistributed on S_n.*

Conjecture 3.2. *The two bistatistics $(\mathbf{des}_2, \mathbf{des})$ and $(\mathbf{pex}, \mathbf{exc})$ are equidistributed on S_n.*

It is interesting to remark that $(\mathbf{des}, \mathbf{cyc})$ and $(\mathbf{exc}, \mathbf{cyc})$ are not equidistributed. Indeed, there are 3 permutations in S_3 having $\mathbf{exc} = 1$ and $\mathbf{cyc} = 2$, namely 132, 213, 321, but only 2 permutations with $\mathbf{des} = 1$ and $\mathbf{cyc} = 2$, videlicet 132 and 213. So, if the Conjectures 3.1 and 3.2 are true then their proofs are probably independent.

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